§3.6 Nonhomogeneous Linear DE (Variation of Parameters)

→ Convert the vector DE variation of parameters formula from chapter 7 to work for high order:

Setup: $y'' + py' + qy = g$

[Where homog. eqn. $y'' + py' + qy = 0$ has solution $y = c_1 y_1 + c_2 y_2$]

$$(y')' = y'' = -qy - py' + g$$

$$\begin{bmatrix} x_1 = y \\ x_2 = y' \end{bmatrix}$$

$$\underline{x}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

[Where homog. system $\underline{x}' = \begin{bmatrix} 0 & 1 \\ -q & -p \end{bmatrix} \underline{x}$ has solution $\underline{x} = c_1 \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + c_2 \begin{bmatrix} y_2 \\ y_2' \end{bmatrix}$]

$$\Rightarrow \Psi = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$$

$$\boxed{\det \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = y_1 y_2' - y_2 y_1' = W(y_1, y_2) \text{ the "\underline{Wronskian}"}}$$

Variation of Parameters Formula (Ch. 7)

$$\underline{x} = c_1 \begin{bmatrix} y_1 \\ y_1' \end{bmatrix} + c_2 \begin{bmatrix} y_2 \\ y_2' \end{bmatrix} + \Psi \left(\int \Psi^{-1} \cdot \begin{bmatrix} 0 \\ g \end{bmatrix} \right)$$

→ y is the top row of this.

$$\Psi^{-1} \cdot \begin{bmatrix} 0 \\ g \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix} = \frac{1}{W(y_1, y_2)} \begin{bmatrix} -y_2 \cdot g \\ y_1 \cdot g \end{bmatrix}$$

Explain with Cramer's Rule instead??

Variation of Parameters Formula:

$y'' + py' + qy = g$ has general solution

$$y = c_1 y_1 + c_2 y_2 + \left(\int \frac{y_1 \cdot g}{W} dt \right) y_2 - \left(\int \frac{y_2 \cdot g}{W} dt \right) y_1$$

EX: Solve $y'' - 2y' - 8y = e^{4t} = g(t)$

Fundamental Solutions

$$y'' - 2y' - 8y = 0$$

$\Downarrow$ char. eqn.

$$r^2 - 2r - 8 = 0$$

$$(r-4)(r+2) = 0 \qquad r = -2, 4$$

$$y_1 = e^{-2t} \quad \text{and} \quad y_2 = e^{4t}$$

Variation of Parameters

$$W(e^{-2t}, e^{4t}) = (e^{-2t})(e^{4t})' - (e^{4t})(e^{-2t})'$$

$$= (e^{-2t})(4e^{4t}) - (e^{4t})(-2e^{-2t})$$

$$= 6e^{2t}$$

$$\int \frac{y_1 \cdot g}{W}\, dt = \int \frac{e^{-2t} \cdot e^{4t}}{6e^{2t}}\, dt$$

$$= \int \tfrac{1}{6}\, dt = \tfrac{1}{6} t$$

$$\int \frac{y_2 \cdot g}{W}\, dt = \int \frac{e^{4t} \cdot e^{4t}}{6e^{2t}}\, dt$$

$$= \int \tfrac{1}{6} e^{6t}\, dt = \tfrac{1}{36} e^{6t}$$

Answer

$$\boxed{y = c_1 e^{-2t} + c_2 e^{4t} + (\tfrac{1}{6} t) e^{4t} - (\tfrac{1}{36} e^{6t}) e^{-2t}}$$

EX: Solve $y'' - 2y' + 5y = e^{t} = g(t)$

Fundamental Solutions

$$r^2 - 2r + 5 = 0$$

$$(r-1)^2 + 4 = 0 \qquad r = 1 \pm 2i$$

$$y_1 = e^t \cos 2t \quad \text{and} \quad y_2 = e^t \sin 2t$$

Variation of Parameters

$$W = (e^t \cos 2t)(e^t \sin 2t)' - (e^t \sin 2t)(e^t \cos 2t)'$$

$$= (e^t \cos 2t)(\cancel{e^t \sin 2t} + 2e^t \cos 2t)$$

$$- (e^t \sin 2t)(\cancel{e^t \cos 2t} - 2e^t \sin 2t)$$

$$= 2e^{2t}(\cos^2 2t + \sin^2 2t) = 2e^{2t}$$

$$\int \frac{y_1 \cdot g}{W}\, dt = \int \frac{e^t \cos 2t \cdot e^t}{2e^{2t}}\, dt$$

$$= \int \tfrac{1}{2} \cos 2t\, dt = \tfrac{1}{4} \sin 2t$$

$$\int \frac{y_2 \cdot g}{W}\, dt = \int \frac{e^t \sin 2t \cdot e^t}{2e^{2t}}\, dt$$

$$= \int \tfrac{1}{2} \sin 2t\, dt = -\tfrac{1}{4} \cos 2t$$

Answer

$$y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + (\tfrac{1}{4} \sin 2t) e^t \sin 2t - (\tfrac{1}{4} \cos 2t) e^t \cos 2t$$

$$\tfrac{1}{4} e^t (\sin^2 2t + \cos^2 2t)$$

$$\boxed{y = c_1 e^t \cos 2t + c_2 e^t \sin 2t + \tfrac{1}{4} e^t}$$

EX: Solve $y'' + 4y' + 4y = t + 1$

Fundamental Solutions

$$r^2 + 4r + 4 = 0$$

$$(r+2)^2 = 0 \qquad r = -2, -2$$

$$y_1 = e^{-2t} \quad \text{and} \quad y_2 = te^{-2t}$$

Variation of Parameters

$$W(e^{-2t}, te^{-2t}) = (e^{-2t})(te^{-2t})' - (te^{-2t})(e^{-2t})'$$

$$= e^{-2t}(e^{-2t} \cancel{- 2te^{-2t}}) \cancel{- te^{-2t}(-2e^{-2t})}$$

$$= e^{-4t}$$

$$\int \frac{y_1 \cdot g}{W}\,dt = \int \frac{e^{-2t}(t+1)}{e^{-4t}}\,dt$$

$$= \int e^{2t}(t+1)\,dt = \tfrac{1}{2}te^{2t} + \tfrac{1}{4}e^{2t}$$

$$\int \frac{y_2 \cdot g}{W}\,dt = \int \frac{te^{-2t}(t+1)}{e^{-4t}}\,dt$$

$$= \int e^{2t}(t^2 + t)\,dt = \tfrac{1}{2}t^2e^{2t}$$

Answer

$$y = c_1e^{-2t} + c_2te^{-2t} + (\cancel{\tfrac{1}{2}te^{2t}} + \tfrac{1}{4}e^{2t})te^{-2t}$$

$$- (\cancel{\tfrac{1}{2}t^2}e^{2t})e^{-2t}$$

(EX continued)

$$\boxed{y = c_1e^{-2t} + c_2te^{-2t} + t/4}$$

Variation of Parameters usually involves a lot of cancelation and simplification

→ For example $\frac{1}{W} = \frac{1}{y_1y_2' - y_2y_1'}$ seems like a scary thing to integrate — but usually it simplifies to be easy

Real, distinct roots

$$W(e^{at}, e^{bt}) = (e^{at})(e^{bt})' - (e^{bt})(e^{at})'$$

$$= be^{(a+b)t} - ae^{(a+b)t}$$

$$= (b-a)e^{(a+b)t}$$

Repeated roots

$$W(e^{at}, te^{at}) = (e^{at})(te^{at})' - (te^{at})(e^{at})'$$

$$= e^{at}(e^{at} \cancel{+ ate^{at}}) - \cancel{te^{at}(ae^{at})}$$

$$= e^{2at}$$

Complex Roots

$$W(e^{at}\cos bt,\ e^{at}\sin bt)$$
$$= (e^{at}\cos bt)(e^{at}\sin bt)' - (e^{at}\sin bt)(e^{at}\cos bt)'$$
$$= e^{at}\cos bt\,(\cancel{ae^{at}\sin bt} + be^{at}\cos bt) - e^{at}\sin bt\,(\cancel{ae^{at}\cos bt} - be^{at}\sin bt)$$
$$= be^{2at}(\cos^2 bt + \sin^2 bt) = \underline{\underline{be^{2at}}}$$

→ ALWAYS simplify W **BEFORE** you try to integrate!

Initial Value Problem:

Do not solve for c_1 & c_2 until **AFTER** completing variation of parameters.

EX: Solve $4y'' - y = t + 1$ with $\begin{cases} y(0) = 1 \\ y'(0) = 2 \end{cases}$

Fundamentel Solutions

$$4r^2 - 1 = 0$$
$$r^2 = 1/4 \qquad \underline{r = \pm 1/2}$$
$$\underline{y_1 = e^{-1/2 t}} \quad \text{and} \quad \underline{y_2 = e^{1/2 t}}$$

(EX continued)

Homogeneous equation

$$4y'' - y = 0$$

has solution

$$y = c_1 e^{-1/2 t} + c_2 e^{1/2 t}$$

→ plugging in initial values would give $\boxed{c_1 = -3/2}$ $\boxed{c_2 = 5/2}$

These are NOT the correct constants for non homogeneous equation!!!

Variation of Parameters

$$W(e^{-1/2 t}, e^{1/2 t}) = (e^{-1/2 t})(e^{1/2 t})' - (e^{1/2 t})(e^{-1/2 t})'$$
$$= 1/2 + 1/2 = \underline{\underline{1}}$$

DE is $y'' - 1/4\,y = 1/4\,(t+1) = g(t)$

$$\int \frac{y_1 g}{W}\,dt = \int e^{-1/2 t} \cdot 1/4\,(t+1)\,dt$$
$$= \underline{\underline{-1/2\, t e^{-1/2 t} - 3/2\, e^{-1/2 t}}}$$

$$\int \frac{y_2 g}{W}\,dt = \int e^{1/2 t} \cdot 1/4\,(t+1)\,dt$$
$$= \underline{\underline{1/2\, t e^{1/2 t} + 3/2\, e^{1/2 t}}}$$

(EX continued)

General Soln

$$y = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} + \left(-\tfrac{1}{2} t e^{-\frac{1}{2}t} - \tfrac{3}{2} e^{-\frac{1}{2}t}\right) e^{\frac{1}{2}t} - \left(\tfrac{1}{2} t e^{\frac{1}{2}t} + \tfrac{3}{2} e^{\frac{1}{2}t}\right) e^{-\frac{1}{2}t}$$

$$y = c_1 e^{-\frac{1}{2}t} + c_2 e^{\frac{1}{2}t} - t - 3$$

$$y' = -\tfrac{1}{2} c_1 e^{-\frac{1}{2}t} + \tfrac{1}{2} c_2 e^{\frac{1}{2}t} - 1$$

Initial Values

$$1 = y(0) = c_1 + c_2 - 3$$

$$+\,2\left(2 = y'(0) = -\tfrac{1}{2} c_1 + \tfrac{1}{2} c_2 - 1\right)$$

$$5 = 0 + 2c_2 - 5 \rightsquigarrow \boxed{c_2 = 5}$$

$$1 = c_1 + 5 - 3 \rightsquigarrow \boxed{c_1 = -1}$$

Answer

$$\boxed{y = -e^{-\frac{1}{2}t} + 5e^{\frac{1}{2}t} - t - 3}$$

Variation of Parameters is a lot of work, but it even works for equations which have non-constant coefficients!

(If you can get y_1 & y_2...)

EX: Solve $t^2 y'' - 2y = 3t^2 - 1$ if $y_1 = t^2$ & $y_2 = 1/t$ (5)

Note: You cannot use the characteristic eqn

$$t^2 r^2 - 2 = 0$$

to solve for y_1 & y_2.

The characteristic equation is not magic.

Variation of Parameters

$$W(t^2, 1/t) = (t^2)(1/t)' - (1/t)(t^2)'$$

$$= t^2(-1/t^2) - (1/t)(2t) = \underline{\underline{-3}}$$

DE is $y'' - \frac{2}{t^2} y = \frac{3t^2 - 1}{t^2} = g(t)$

$$\int \frac{y_1 \cdot g}{W} dt = \int -\tfrac{1}{3}\left(t^2 \cdot (3 - 1/t^2)\right) dt$$

$$= -\tfrac{1}{3} t^3 + \tfrac{1}{3} t$$

$$\int \frac{y_2 \cdot g}{W} dt = \int -\tfrac{1}{3}\left(1/t \cdot (3 - 1/t^2)\right) dt$$

$$= -\ln t - \tfrac{1}{6} \tfrac{1}{t^2}$$

Answer

$$y = c_1 t^2 + c_2 \tfrac{1}{t} + \left(-\tfrac{1}{3} t^3 + \tfrac{1}{3} t\right) \tfrac{1}{t} - \left(-\ln t - \tfrac{1}{6} \tfrac{1}{t^2}\right) t^2$$

(EX continued)

$$\boxed{y = c_1 t^2 + c_2 \tfrac{1}{t} + (\ln t - \tfrac{1}{3})t^2 + \tfrac{1}{2}}$$

If there is more time:

EX: Solve $t^2 y'' - 3ty' + 4y = t^2 \ln t$

if $y_1 = t^2$ & $y_2 = t^2 \ln t$

Variation of Parameters

$$W(t^2, t^2\ln t) = (t^2)(t^2 \ln t)' - (t^2 \ln t)(t^2)'$$

$$= t^2(\cancel{2t\ln t} + t) - \cancel{t^2 \ln t\,(2t)}$$

$$= \underline{t^3}$$

DE is $y'' - \tfrac{3}{t}y' + \tfrac{4}{t^2}y = \ln t = g(t)$

$$\int \frac{y_1 \cdot g}{W}dt = \int \frac{t^2 \ln t}{t^3}dt = \int \frac{\ln t}{t}dt = \underline{\tfrac{1}{2}(\ln t)^2}$$

$$\int \frac{y_2 \cdot g}{W}dt = \int \frac{t^2 \ln t \cdot \ln t}{t^3}dt = \int \frac{(\ln t)^2}{t}dt = \underline{\tfrac{1}{3}(\ln t)^3}$$

Answer

$$\boxed{y = c_1 t^2 + c_2 t^2 \ln t + \tfrac{1}{6}t^2(\ln t)^3}$$

The two previous problems had the form $a\underline{t^2}y'' + b\underline{t}y' + cy$

Equations like this are called "Euler Equations"

They often have solutions like

$$\underline{y = t^n}$$

→ Plug in and solve for n:

$$at^2y'' + bty' + cy = 0$$

$$at^2(t^n)'' + bt(t^n)' + c(t^n) = 0$$

vertical asymptote at $t=0$??

$$t^n(a(n)(n-1) + b(n) + c) = 0$$

$$an^2 + (b-a)n + c = 0$$

EX: $t^2y'' - 2y = 0$

$y = t^n$

$$t^2 \cdot (n(n-1)t^{n-2}) - 2t^n = 0$$

$$t^n(n^2 - n - 2) = 0$$

$$(n-2)(n+1) = 0 \qquad n = 2, -1$$

$\underline{y_1 = t^2}$ and $\underline{y_2 = t^{-1} = 1/t}$

If there is more time:

→ What about higher order DE?

Variation of parameters is still possible, but *much* messier...

EX 3rd order

$$y''' + p_1 y'' + p_2 y' + p_3 y = g$$

with fundamental solutions y_1, y_2, & y_3.

Vector equation is $\begin{bmatrix} x_1 = y \\ x_2 = y' \\ x_3 = y'' \end{bmatrix}$

$$\underline{x}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -p_3 & -p_2 & -p_1 \end{bmatrix} \underline{x} + \begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$$

Homogeneous equation has solution

$$\underline{x} = c_1 \begin{bmatrix} y_1 \\ y_1' \\ y_1'' \end{bmatrix} + c_2 \begin{bmatrix} y_2 \\ y_2' \\ y_2'' \end{bmatrix} + c_3 \begin{bmatrix} y_3 \\ y_3' \\ y_3'' \end{bmatrix}$$

$$\Rightarrow \Psi = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix}$$

Wronskian $W(y_1, y_2, y_3) = \det \Psi$

this has 6 terms — one for each way of choosing who gets $\frac{d}{dt}$ & $\frac{d^2}{dt^2}$

(EX continued)

Use Cramer's Rule to get $\Psi^{-1}\begin{bmatrix} 0 \\ 0 \\ g \end{bmatrix}$

$$\frac{\begin{vmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ g & y_2'' & y_3'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} = \frac{\overset{W(y_2, y_3)}{\begin{vmatrix} y_2 & y_3 \\ y_2' & y_3' \end{vmatrix}} \cdot g}{W(y_1, y_2, y_3)}$$

$$\frac{\begin{vmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & g & y_3'' \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} = -\frac{\overset{W(y_1, y_3)}{\begin{vmatrix} y_1 & y_3 \\ y_1' & y_3' \end{vmatrix}} \cdot g}{W(y_1, y_2, y_3)}$$

$$\frac{\begin{vmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & g \end{vmatrix}}{\begin{vmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{vmatrix}} = \frac{\overset{W(y_1, y_2)}{\begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix}} \cdot g}{W(y_1, y_2, y_3)}$$

Integrate these and multiply by y_1, y_2, y_3 to get the Variation of Parameters formula...

Variation of Parameters formula (3rd order)

$$y''' + p_1 y'' + p_2 y' + p_3 y = g$$

has general solution

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$+ \left(\int \frac{W(y_1, y_2) \cdot g}{W(y_1, y_2, y_3)} \, dt \right) \cdot y_3$$

$$- \left(\int \frac{W(y_1, y_3) \cdot g}{W(y_1, y_2, y_3)} \, dt \right) \cdot y_2$$

$$+ \left(\int \frac{W(y_2, y_3) \cdot g}{W(y_1, y_2, y_3)} \, dt \right) \cdot y_1$$

(This is beautiful and terrible at the same time.)